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Trapped waves in the neighbourhood of a sonic-type singularity

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A model equation is derived to study trapped nonlinear waves with a turning effect, occurring in disturbances induced on a two-dimensional steady flow. Only unimodal disturbances under the short wave assumption are considered, when the wave front of the induced disturbance is plane. In the neighbourhood of certain special points of sonic-type singularity, the disturbances are governed by a single first-order partial differential equation in two independent variables. The equation depends on the steady flow through three parameters, which are determined by the variations of velocity and depth, for example (in the case of long surface water waves), along and perpendicular to the wave front. These parameters help us to examine various relative effects. The presence of shocks in a continuously accelerating or decelerating flow has been studied in detail.

1. Introduction

Nonlinear hyperbolic waves in many dimensions have been the subject of many recent investigations. Of special significance is the case when unsteady unimodal disturbance waves are generated in a basic two-dimensional steady flow and are momentarily trapped at certain points in the flow. This occurrence is of interest, for example, in the transonic flow of a polytropic gas past an aerofoil or through converging or diverging nozzles. This could also occur in the case of long surface water waves over a sloping bottom. The case of transonic flow has been studied in great detail by Prasad (1973), Spee (1971) and Nieulwand & Spee (1968). The case of water waves has received little attention and it is to this study that we shall devote this paper.

In the above two cases, there are certain effects which play an important part in determining the nature of the flow and in the existence of continuously accelerating and decelerating flows through the speed of sound; these are tested below.

(1) The steepening of the waves, due to the nonlinearity present in the system.

(2) The trapping of waves, due to the presence of points in the flow field at which all the components of the ray velocity approach zero. At these points, which are singularities of the sonic type, trapping occurs only when the normal to the wave front is in a privileged direction, namely in the direction of the streamlines of the steady flow at that point.

(3) The turning of the wave front, due to the presence of velocity gradients along the wave front, namely in a direction perpendicular to the streamlines. This turning of the wave front contributes to its eventual release from the trapped position.

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(4) The growth or decay of disturbances, due to inhomogeneity, such as favourable or adverse inclination of the bottom in surface water waves.

These four effects, including their interaction, are investigated here in detail. The occurrence of shocks in continuously decelerating flows through a sonic-type singularity, their velocity of propagation, their growth or decay are dictated by the abovementioned effects.

In this paper, we derive an approximate equation, which governs the propagation of disturbance pulses moving slowly over a two-dimensional steady flow of long surface water waves in shallow water over a sloping bottom. However, the model equation derived here is very general and applicable to all hyperbolic genuinely nonlinear waves with turning effect in two space dimensions (including the transonic problem). Turning of the waves in transonic flow has been discussed by Prasad & Krishnan (1977) but the approximate model equation obtained by them is very complicated due to the presence of the curvature of the wave front. In our analysis, we have assumed the wave front of the induced disturbance to be plane. This is not a major assumption, as the curvature can be combined with damping or growth of the disturbance due to the convergence or divergence of the rays; however the simplification in the analysis is considerable and the resulting model can be studied analytically. We restrict our attention to those disturbances which are confined in the neighbourhood of a wave front (short-wave approximation), especially around singularities of the sonic type, where the normal to the wave front is in the direction of the stream line. These disturbances are governed by a single first-order partial differential equation in two independent variables (cf. Prasad (1973) in the case of transonic flows without turning effects considered), which lends itself to a detailed study. The equation depends on the steady flow through three parameters, which depend on the variations of velocity and depth both along and perpendicular to the wave front. These parameters help us to study the relative effects of turning of the wave front and increasing or decreasing depth of the bottom. The area of the disturbance in the phase plane associated with the characteristic equations of the governing partial differential equation grows or decays with time depending on whether the depth is decreasing or increasing normal to the wave front. The presence of shocks in a continuously accelerating or decelerating flow has been studied in detail.

2. Equations governing the motion and derivation of the model equation

Let u_0, v_0, a_0, η_0 all of which are functions of x and y describe the steady flow of a fluid in shallow water over a sloping bottom. Here $q = [u_0, v_0]$ is the fluid velocity in the (x, y) plane perpendicular to the height, η_0 is the height of the fluid over the still water level and $a_0 = (\eta_0 + h)^{\frac{1}{2}}$ is the undisturbed sound velocity in the fluid.

The equations satisfied by u_0, v_0 and η_0 in the long wavelength limit are (in terms of non-dimensional quantities):

$$u_0 u_{0x} + v_0 u_{0y} + \eta_{0x} = 0, \tag{1}$$

$$u_0 v_{0x} + v_0 v_{0y} + \eta_{0y} = 0 \tag{2}$$

and

$$u_0 \eta_{0x} + v_0 \eta_{0y} + (\eta_0 + h)(u_{0x} + v_{0y}) = -u_0 h_x - v_0 h_y. \tag{3}$$

Let

$$\left.\begin{aligned} u &= u_0 + u_1(x,y,t), \quad \eta = \eta_0 + \eta_1(x,y,t),\\ v &= v_0 + v_1(x,y,t), \quad a = a_0 + a_1(x,y,t)\\ &\qquad\qquad = (\eta_0 + h)^{\frac{1}{2}} + \frac{\eta_1(x,y,t)}{2a_0} \end{aligned}\right\} \tag{4}$$

denote a perturbation of the steady flow, where the quantities with subscript 1 are of an order of magnitude, say δ, smaller than those with subscript 0. The above quantities give a first-order approximation to the flow. Neglecting terms of $O(\delta^2)$, the system of equations in matrix form governing u_1, v_1 and η_1 is found to be:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} = \mathbf{C}, \tag{5}$$

where

$$\left.\begin{aligned} \mathbf{U} &= \begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u_0+u_1 & 0 & 1 \\ 0 & u_0+u_1 & 0 \\ \eta_0+\eta_1+h & 0 & u_0+u_1 \end{bmatrix},\\ \mathbf{B} &= \begin{bmatrix} v_0+v_1 & 0 & 0 \\ 0 & v_0+v_1 & 1 \\ 0 & \eta_0+\eta_1+h & v_0+v_1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -u_1u_{0x} - v_1u_{0y} \\ -u_1v_{0x} - v_1v_{0y} \\ -u_1h_x - v_1h_y - u_1\eta_{0x} - v_1\eta_{0y} \\ -\eta_1(u_{0x}+v_{0y}) \end{bmatrix}. \end{aligned}\right\} \tag{6}$$

The above system is hyperbolic with t as a time-like variable. If n_1, n_2 represent the direction cosines of a normal to the wave front at t = constant, then the characteristic velocities of the system are distinct and are given by

$$c_1 = un_1 + vn_2, \quad c_{2,3} = un_1 + vn_2 \mp a. \tag{7}$$

We restrict ourselves in this study to disturbances which are confined in the neighbourhood of a particular characteristic surface, say, $\phi(x,y,t) = 0$ given by

$$\phi_t + u\phi_x + v\phi_y + a(\phi_x^2 + \phi_y^2)^{\frac{1}{2}} = 0. \tag{8}$$

We note that n_1 and n_2 are given by

$$n_1 = \frac{\phi_x}{(\phi_x^2+\phi_y^2)^{\frac{1}{2}}}, \quad n_2 = \frac{\phi_y}{(\phi_x^2+\phi_y^2)^{\frac{1}{2}}} \tag{9}$$

and satisfy $n_1^2 + n_2^2 = 1$. In terms of the angle θ, which the normal to the wave front makes with the positive x-axis

$$n_1 = \cos\theta, \quad n_2 = \sin\theta. \tag{10}$$

The equation to a bicharacteristic curve on this surface is given by (Prasad 1975)

$$x = x(\sigma_1), \quad y = y(\sigma_1), \quad t = t(\sigma_1), \tag{11}$$

where

$$\frac{dx}{d\sigma_1} = u + n_1 a, \quad \frac{dy}{d\sigma_1} = v + n_2 a, \quad \frac{dt}{d\sigma_1} = 1,$$

$$\frac{d\theta}{d\sigma_1} = n_1(n_2u_x - n_1u_y) + n_2(n_2v_x - n_1v_y) + (n_2a_x - n_1a_y)$$

or equivalently

$$\frac{dn_1}{d\sigma_1} = -n_2\frac{d\theta}{d\sigma_1}, \quad \frac{dn_2}{d\sigma_1} = n_1\frac{d\theta}{d\sigma_1}. \tag{12}$$

Equation (12) helps to determine n_1, n_2 as functions of x and y, when they are specified initially at $t = 0$. We introduce a set of new variables ω_1, ω_2, ω_3 by the relation

$$\begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} n_1 & n_1 & -n_2 \\ n_2 & n_2 & n_1 \\ a & -a & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \tag{13}$$

where the columns in the matrix of transformation is the set of linearly independent right eigenvectors corresponding to the characteristic velocities c_3, c_2 and c_1, respectively. Here we are interested only in disturbances which consist of a single mode confined in the neighbourhood of a characteristic surface (or a moving wave front with velocity c_3). We further make the short wave assumption, namely that the disturbance is non-zero over a distance of order δ from the wave front, where δ is small compared to the characteristic length of the problem. Following Prasad (1975), we can show that in the neighbourhood of the characteristic surface (8),

$$\omega_2, \omega_3 \ll \omega_1 = O(\delta). \tag{14}$$

The system of equations (5) together with the transformation (11) reduces to the following equation in the neighbourhood of $\phi(x,y,t) = 0$, under the short-wave assumption (14):

$$\frac{d\omega_1}{d\sigma_0} + \tfrac{3}{2}\omega_1 \left(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}\right) \omega_1 = K\omega_1, \tag{15}$$

where

$$\frac{d}{d\sigma_0} = \frac{\partial}{\partial t} + (u_0 + a_0 n_1)\frac{\partial}{\partial x} + (v_0 + a_0 n_2)\frac{\partial}{\partial y} \tag{16}$$

and

$$K = -\frac{a_0}{2}(n_{1x} + n_{2y}) - \frac{1}{2a_0}\left[(u_0 + a_0 n_1)\frac{\partial a_0}{\partial x} + (v_0 + a_0 n_2)\frac{\partial a_0}{\partial y}\right] - \tfrac{1}{2}(1 + n_1^2)\, u_{0x}$$
$$- \frac{n_1 n_2}{2}(u_{0y} + v_{0x}) - \tfrac{1}{2}(1 + n_2^2)\, v_{0y} - \frac{1}{2a_0}[n_1 h_x + n_2 h_y + n_1 \eta_{0x} + n_2 \eta_{0y}]. \tag{17}$$

$d/d\sigma_0$ denotes derivation along a zeroth-order bicharacteristic of the system (5) given by equations (11) and (12).

In particular, we now concentrate our attention on the study of the motion of the wave in the neighbourhood of certain special points, namely those at which both components of the ray (or bicharacteristic) velocity vanish. At these points, denoted by (x^*, y^*), we have

$$u_0^* + a_0^* n_1 = 0, \quad v_0^* + a_0^* n_2 = 0 \tag{18a}$$

($*$ above a quantity denoting its value at the point (x^*, y^*)), so that the normal to the wave front is in the direction of the streamline at that point, i.e.

$$n_1 = -\frac{u_0^*}{a_0^*}, \quad n_2 = -\frac{v_0^*}{a_0^*} \quad \text{and} \quad u_0^{*2} + v_0^{*2} = a_0^{*2}. \tag{18b}$$

From equation (15) we notice that at those points (x^*, y^*), when n_1 and n_2 are given by (18b), the velocity of propagation of the disturbance approaches zero and the wave is temporarily trapped. Only these waves will be trapped at sonic points when the wave fronts are normal to the streamlines. To study these trapped waves, when the wave front is assumed plane, we transform to new variables:

$$\left.\begin{aligned} \xi &= n_1(t)\,(x-x^*)+n_2(t)\,(y-y^*), \\ \zeta &= -n_2(t)\,(x-x^*)+n_1(t)\,(y-y^*), \\ t' &= t, \end{aligned}\right\} \tag{19}$$

where ξ, ζ are co-ordinates perpendicular to and along the wave front. If, at $t = 0$, $n_1(0)$ and $n_2(0)$ are given by (18*b*), then at subsequent times $n_1(t)$, $n_2(t)$ denote the rotation of the wave front about the original trapped configuration. As we are confining ourselves to the neighbourhood of special points (x^*, y^*) specified by (18*a*), we can simplify equation (15) by setting

$$\left.\begin{aligned} u_0 &= u_0^* + \xi u_{0\xi}^* + \zeta u_{0\zeta}^*, \\ v_0 &= v_0^* + \xi v_{0\xi}^* + \zeta v_{0\zeta}^*, \\ a_0 &= a_0^* + \xi a_{0\xi}^* + \zeta a_{0\zeta}^*, \end{aligned}\right\} \tag{20}$$

where terms of $O(\xi^2)$ and $O(\zeta^2)$ have been omitted. Also for small times t,

$$\left.\begin{aligned} n_1(t) &= n_1(0) + tn_1'(0) + \frac{t^2}{2!}n_1''(0), \\ n_2(t) &= n_2(0) + tn_2'(0) + \frac{t^2}{2!}n_2''(0), \end{aligned}\right\} \tag{21}$$

where terms of $O(t^3)$ have been omitted. The reason for retaining terms up to $O(t^2)$ in (21) and only up to $O(\xi, \zeta)$ in (20) can be justified in the following way.

The rotation of the wave front about the trapped configuration is specified by equations (12). In the neighbourhood of (x^*, y^*), we get

$$\left(\frac{d\theta}{dt}\right)^* = \frac{u_0^* u_{0\zeta}^* + v_0^* v_{0\zeta}^* - a_0^* a_{0\zeta}^*}{a_0^*}. \tag{22}$$

Therefore $\theta - \theta_0^* = O(t)$ for small values of t, where θ_0^* is the undisturbed value of θ at (x^*, y^*). However,

$$\left.\begin{aligned} \left(\frac{dx}{dt}\right)^* &= u_0^* + a_0^* n_1 = 0 + a_0^* t n_1'(0), \\ \left(\frac{dy}{dt}\right)^* &= a_0^* t n_2'(0), \end{aligned}\right\} \tag{23}$$

and this gives us

$$x - x^* = O(t^2), \quad y - y^* = O(t^2), \tag{24}$$

so that $(x-x^*)$ and $(y-y^*)$ are of $O(t^2)$ for small values of t. So, if we retain terms up to $O(\xi)$ in (20), we shall have to retain terms up to $O(t^2)$ in (21), where

$$t^2 = O(\xi). \tag{25}$$

Using equations (20) and (21), equation (15) simplifies to

$$\begin{aligned} \frac{\partial \omega_1}{\partial t'} &+ \left[\tfrac{3}{2}\omega_1 + \xi\left(\frac{-u_0^* u_{0\xi}^* - v_0^* v_{0\xi}^* + a_0^* a_{0\xi}^*}{a_0^*}\right) + \frac{t^2}{2}\left(n_1''(0)\,u_0^* + n_2''(0)\,v_0^*\right)\right]\frac{\partial \omega_1}{\partial \xi} \\ &+ \Bigg[t(u_0^* u_{0\zeta}^* + v_0^* v_{0\zeta}^* - a_0^* a_{0\zeta}^*) + \frac{t^2}{2}\left(-n_2''(0)\,u_0^* + n_1''(0)\,v_0^*\right) \\ &+ \frac{\xi}{a_0^*}\{-u_0^*(v_{0\xi}^* + u_{0\zeta}^*) - v_0^*(v_{0\zeta}^* - u_{0\xi}^*) + a_0^* a_{0\zeta}^*\} + \frac{\zeta}{a_0}(v_0^* u_{0\zeta}^* - u_0^* v_{0\zeta}^*)\Bigg]\frac{\partial \omega_1}{\partial \zeta} = K\omega_1. \end{aligned} \tag{26}$$

Under the short-wave assumption, we require that the variation of ω_1 with ξ, ζ and t' are such that $\partial\omega_1/\partial\xi$ is the most prominent. This implies that ω_1 varies more rapidly perpendicular to the plane wave front than along it. Other quantities being of comparable order, we neglect terms containing $\partial\omega_1/\partial\zeta$ in comparison with $\partial\omega_1/\partial\xi$. Suitably scaling ω_1, ξ and t' and neglecting higher-order terms, ω_1 is then governed by

$$\frac{\partial\bar{\omega}_1}{\partial\bar{t}'_1}+\left[\tfrac{3}{2}\bar{\omega}_1+\bar{\xi}\left(\frac{-u_0^*u_{0\xi}^*-v_0^*v_{0\xi}^*+a_0^*a_{0\xi}^*}{a_0^*}\right)+\frac{\bar{t}'^2}{2}(n_1''(0)\,u_0^*+n_2''(0)\,v_0^*)\right]\frac{\partial\bar{\omega}_1}{\partial\bar{\xi}}=K\bar{\omega}_1, \quad (27)$$

where all barred quantities are of order one. The derivatives of n_1 and n_2 can be calculated using the ray equations (12). After some lengthy computation, we obtain

$$\frac{\partial\bar{\omega}_1}{\partial\bar{t}'}+[\tfrac{3}{2}\bar{\omega}_1+\bar{C}_\xi\bar{\xi}+\bar{C}_t\bar{t}'^2]\frac{\partial\omega_1}{\partial\bar{\xi}}=\bar{K}\bar{\omega}_1, \quad (28)$$

where

$$\left.\begin{aligned}\bar{C}_\xi &= \frac{-u_0^*u_{0\xi}^*-v_0^*v_{0\xi}^*+a_0^*a_{0\xi}^*}{a_0^*} = -\frac{3}{2}\frac{\partial q_0^*}{\partial\xi}+\frac{h_\xi^*}{2a_0^*},\\ \bar{C}_t &= \frac{1}{2a_0^*}(u_0^*u_{0\zeta}^*+v_0^*v_{0\zeta}^*-a_0^*a_{0\zeta}^*)^2,\\ \bar{K} &= -\bar{C}_\xi-\frac{h_\xi^*}{2a_0^*}=\frac{3}{2}\frac{\partial q_0^*}{\partial\xi}-\frac{h_\xi^*}{a_0^*},\end{aligned}\right\} \quad (29)$$

where $\partial q_0^*/\partial\xi$ represents the space rate of change of the fluid speed at the point (x^*, y^*) as we move along the streamline in the steady solution, and is equal to the acceleration of the fluid element divided by a_0^*.

We observe the following:

(1) When the variation in depth in the direction of the streamline (i.e. perpendicular to the wave front) is zero, then $\bar{K} = -\bar{C}_\xi$ and the number of parameters is reduced. This is observed in the case of transonic flow by Prasad (1973).

(2) The turning effect of the wave front is represented by the parameter $\bar{C}_t$, which depends on the gradient of the steady flow perpendicular to the streamlines. If this were zero (i.e. the quantities in steady flow did not vary along the wave front), then there is no turning of the wave front.

3. Solution and discussion of results

Equation (28) is a single first-order partial differential equation in two independent variables and its solution $\bar{\omega}_1(\bar{\xi}, \bar{t}')$ satisfying suitable initial conditions can be obtained easily. Writing the characteristic equations associated with equation (28), we can solve for $\bar{\omega}_1$ and $\bar{\xi}$ in terms of $\bar{t}'$ and two parameters $\bar{\omega}_{10}$ and $\bar{\xi}_0$. $\bar{\omega}_{10}$ represents the value of $\bar{\omega}_1$ at the point $(\bar{\xi}_0, 0)$ on the $\bar{\xi}$-axis in the $(\bar{\xi}, \bar{t}')$ plane where the characteristic curve through $(\bar{\xi}, \bar{t}')$ meets the initial line $\bar{t}' = 0$. Omitting bars, we have

$$\left.\begin{aligned}\omega_1 &= \omega_{10}\,e^{Kt'},\\ \xi &= \xi_0 e^{C_\xi t'}+\frac{3}{2}\frac{\omega_{10}}{K-C_\xi}(e^{Kt'}-e^{C_\xi t'})-\frac{C_t}{C_\xi^3}(C_\xi^2t'^2+2C_\xi t'-2+2e^{C_\xi t'}).\end{aligned}\right\} \quad (30)$$

The basic undisturbed solution is represented in the (ω_1, ξ) plane by the line $\omega_1 = 0$. We consider perturbations bounded in space, namely those represented in the (ω_1, ξ) plane by a closed curve, a part of whose boundary is the line $\omega_1 = 0$. In a perturbation, the space rate of change of ω_1 as we move with wave velocity $(\frac{3}{2}\omega_1+\bar{C}_\xi\xi+C_t t'^2)$ is $K\omega_1/(\frac{3}{2}\omega_1+C_\xi\xi+C_t t'^2)$. If S is the area bounded by an arbitrary closed curve in the (ω_1, ξ) plane, whose points move in accordance with equation (28), then

$$\frac{1}{S}\frac{dS}{dt} = \frac{\partial}{\partial\xi}(\tfrac{3}{2}\omega_1+C_\xi\xi+C_t t'^2)+\frac{\partial}{\partial\omega_1}(K\omega_1) = C_\xi+K = -\frac{h^*_\xi}{2a^*_0}. \tag{31}$$

If $h^*_\xi/2a^*_0 < 0$, the disturbed area grows exponentially with time, whereas for $h^*_\xi/2a^*_0 > 0$ it decays rapidly. When the depth of the bottom does not vary along the streamline (i.e. $h^*_\xi/2a^*_0 = 0$), the constant area rule applies.

If we prescribe the initial shape of the pulse at $t' = 0$ by the function

$$\omega_1(\xi, 0) = \omega_{10}(\xi_0),$$

then at any other time t', the slope of the pulse can be obtained in terms of the initial slope $d\omega_{10}/d\xi_0$ at the point ξ_0 (i.e. the point on $t' = 0$ through which the characteristic of equation (28) from (ξ, t') passes) by the relation:

$$\frac{\partial\omega_1}{\partial\xi} = \frac{\frac{2}{3}(K-C_\xi)}{1-\exp[-(K-C_\xi)t'][1+\frac{2}{3}(K-C_\xi)/(d\omega_{10}/d\xi_0)]}. \tag{32}$$

Case 1. $K-C_\xi > 0$, i.e. $\partial q^*_0/\partial\xi > h^*_\xi/2a^*_0$, i.e. the acceleration of the fluid element is greater than half the variation of h along the streamline. In this case, if the initial slope $d\omega_{10}/d\xi_0$ were negative, then the slope $\partial\omega_1/\partial\xi$ at time $\bar{T}$ would become infinite, provided

$$\bar{T} = \frac{1}{(K-C_\xi)}\ln\left(1+\frac{2(K-C_\xi)}{3|d\omega_{10}/d\xi_0|}\right).$$

Thereafter the profile would fold. So a shock wave always appears in this case *first* at time T given by

$$T = \bar{T}_{\min} = \frac{1}{(K-C_\xi)}\ln\left(1+\frac{2(K-C_\xi)}{3\,|d\omega_{10}/d\xi_0|_{\max}}\right). \tag{33}$$

If the initial slope were positive, then the denominator in (32) would never vanish and no shock wave will form. As $t \to \infty$,

$$\partial\omega_1/\partial\xi \to \tfrac{2}{3}(K-C_\xi). \tag{34}$$

Case 2. $K-C_\xi < 0$, i.e. $\partial q^*_0/\partial\xi < h^*_\xi/2a^*_0$, i.e. the acceleration of the fluid element is less than half the variation of h along a streamline. In this case

$$\frac{\partial\omega_1}{\partial\xi} = -\frac{\frac{2}{3}(C_\xi-K)}{1-\exp[(C_\xi-K)t'][1+\frac{2}{3}(C_\xi-K)/(d\omega_{10}/d\xi_0)]}. \tag{35}$$

The denominator vanishes only if $d\omega_{10}/d\xi_0$ is a sufficiently large negative quantity, namely

$$\frac{d\omega_{10}}{d\xi_0} < -\frac{2(C_\xi-K)}{3}. \tag{36}$$

Otherwise the slope never becomes infinite, namely no shock is formed. The presence of positive $(C_\xi-K)$ retards the nonlinear steepening leading to a shock and, unless the initial negative slope is sufficiently steep, a shock will not form. As t tends to infinity, the slope tends to zero everywhere.

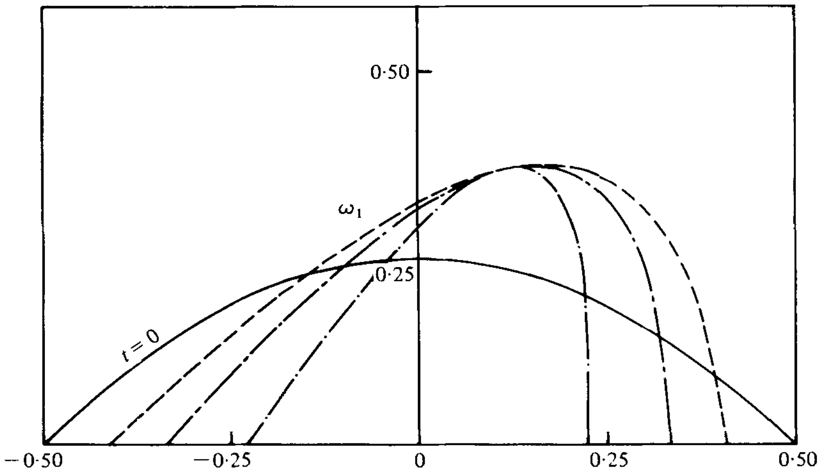

FIGURE 1. A plot of ω_1 *vs.* ξ before the formation of a shock. $\bar{h}_\xi$ positive tends to hasten shock formation and reduce the disturbed area. $\bar{h}_\xi$ negative tends to delay shock formation and increase the disturbed area. ——, $\bar{h}_\xi = 0$, $C_t = 0$; – – –, $\bar{h}_\xi = -\frac{1}{2}$, $C_t = 0$; –·–·–, $\bar{h}_\xi = 1$, $C_t = 0$. $K = 1{\cdot}0$, $t = 0{\cdot}4$.

Case 3. $K - C_\xi = 0$ or $\partial q_0^*/\partial\xi = h_\xi^*/2a_0^*$. In this case a shock will form only if the initial slope is negative. The shock appears first at time T given by

$$T = \frac{2}{3}\left|\frac{d\omega_{10}}{d\xi_0}\right|_{\max}. \tag{37}$$

Once a shock forms, its motion can be followed by using the result that for a weak shock its velocity is the arithmetic mean of characteristic velocities just ahead and just behind the shock. The shock position ξ_s satisfies the equation

$$\frac{d\xi_s}{dt} = C_\xi \xi_s + \tfrac{3}{4}\{\omega_a(\xi_s, t) + \omega_b(\xi_s, t)\}. \tag{38}$$

We have studied case 1 in detail numerically and shown the results with the help of graphs, starting with an initial parabolic profile in the (ω_1, ξ) plane. This initial profile has the maximum of its negative slope at the leading edge, so that shock will first appear at the leading edge. We have drawn the profiles in three cases, namely $\bar{h}_\xi = h_\xi^*/2a_0^* = 0$, $\bar{h}_\xi = -\frac{1}{2}$ and $\bar{h}_\xi = 1$. We have chosen $K = 1$ so $K - C_\xi$ is positive in all three cases. Before the formation of the shock at $t = 0{\cdot}4$ (figure 1), the maximum amplitude in all three cases is almost the same and occurs at nearly the same value of ξ. The effect of non-zero C_t is to shift the graph to the right by an extremely small amount, as can be verified from equation (28). Depending on whether $\bar{h}_\xi$ is positive, negative or zero, the area of the disturbance in (ω_1, ξ) plane decays, grows or remains constant, respectively, with time. The shock appears at the leading edge at an earlier time in the case when $\bar{h}_\xi$ is positive and later when $\bar{h}_\xi$ is negative when compared to that when $\bar{h}_\xi = 0$ (table 1). The presence of C_t has no influence on the time at which the shock begins to form [cf. equation (33)]. After formation, the shock first begins to move backwards and then moves forward with a greater shock velocity when $\bar{h}_\xi$ is negative, than when $\bar{h}_\xi$ is zero (see figure 2). When $\bar{h}_\xi$ is positive, the shock continues to move

	$\bar{h}_\xi$	C_t	Time at which shock forms T	Position at which shock forms ξ_{s_0}	At t = 0·7			At t = 1·0			At t = 1·7		
					Position of shock ξ_s	Base width of disturbance	Amplitude of shock	Position of shock ξ_s	Base width of disturbance	Amplitude of shock	Position of shock ξ_s	Base width of disturbance	Amplitude of shock
(1)	0	0	0·424	0·327	0·296	0·55	0·501	0·327	0·51	0·591	0·453	0·49	0·687
(2)	0	0·1	0·424	0·330	0·305	0·54	0·503	0·351	0·51	0·581	0·748	0·48	0·687
(3)	−0·5	0	0·462	0·397	0·390	0·74	0·483	0·452	0·75	0·647	0·914	1·05	1·03
(4)	−0·5	0·1	0·462	0·400	0·399	0·74	0·487	0·479	0·75	0·640	1·283	1·04	1·01
(5)	1·0	0	0·366	0·240	0·181	0·30	0·483	0·180	0·25	0·466	0·111	0·11	0·240
(6)	1·0	0·1	0·366	0·242	0·189	0·30	0·479	0·200	0·25	0·470	0·317	0·11	0·239

TABLE 1

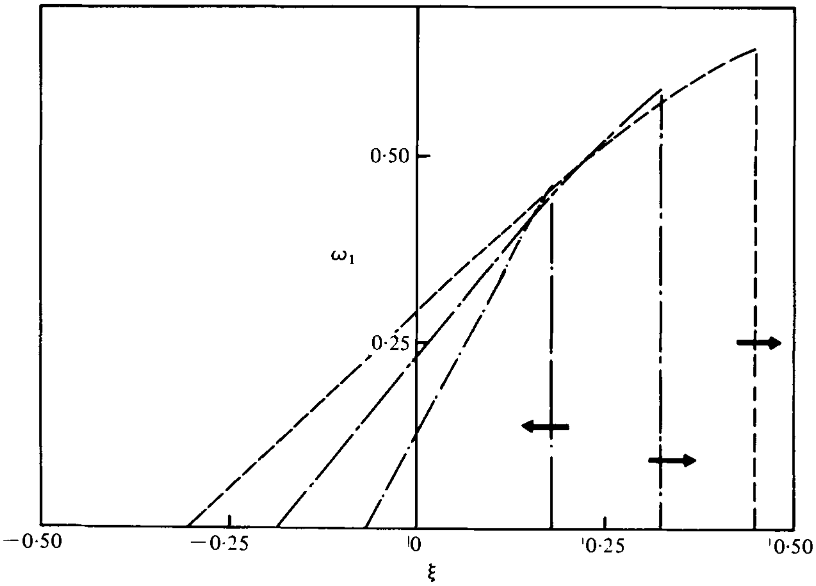

FIGURE 2. A plot of ω_1 *vs.* ξ after the formation of a shock. $\bar{h}_\xi$ positive tends to lessen the amplitude of the shock and causes it to move in the backward direction. $\bar{h}_\xi$ negative tends to increase the amplitude of the shock and causes it to move faster in the forward direction. ——, $\bar{h}_\xi = 0$, $C_t = 0$; – – –, $\bar{h}_\xi = -\frac{1}{2}$, $C_t = 0$; —·—·—, $\bar{h}_\xi = 1$, $C_t = 0$. $K = 1·0$, $t = 1·0$.

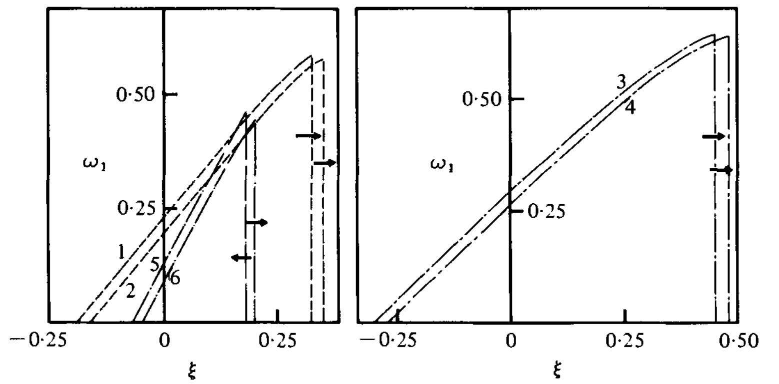

FIGURE 3. The effect of C_t is to cause the shock to move faster in the forward direction. It does not have any appreciable effect on the amplitude of the shock. (1) $\bar{h}_\xi = 0$, $C_t = 0$; (2) $\bar{h}_\xi = 0$, $C_t = 0·1$; (3) $\bar{h}_\xi = -\frac{1}{2}$, $C_t = 0$; (4) $\bar{h}_\xi = -\frac{1}{2}$, $C_t = 0·1$; (5) $\bar{h}_\xi = 1$, $C_t = 0$; (6) $\bar{h}_\xi = 1$, $C_t = 0·1$. $K = 1·0$, $t = 1·0$.

backwards. The amplitude of the shock increases steadily with time when $\bar{h}_\xi$ is negative or zero, but decreases when $\bar{h}_\xi$ is positive. The effect of C_t is to give an added shock velocity in the forward-moving direction, as seen in figures 3 and 4, at times $t = 1·0$ and $t = 1·7$, respectively. Even when $\bar{h}_\xi$ is positive, after a certain time the effect of C_t is to cause the shock to move forward instead of backward. The amplitude and the base width of the triangular form of the shock are hardly affected by the presence of C_t. The area rule of the disturbance can be verified from table 1. At $t = 1·7$, the disturbance in all three cases has taken a triangular form, which will persist for all later times.

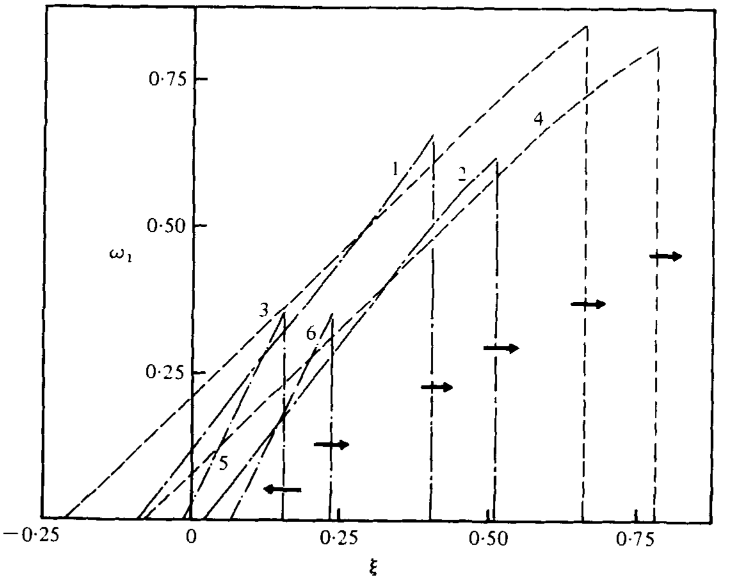

FIGURE 4. The combined effect of $\bar{h}_\xi$ and C_t on the amplitude and velocity of the shock. $K = 1{\cdot}0$, $t = 1{\cdot}7$. For key to numbers see figure 3.

Regarding the nonlinear stability of the flow, we can conclude that, when $\bar{h}_\xi = 0$ and $-\frac{1}{2}$, the flow is unstable, whereas, for $\bar{h}_\xi = 1$, the flow is stable as the disturbance shrinks rapidly with time.

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